HW11C: 1ceg, 2ch, 3, 6bd, 7c, 11

1.) $z = 5 - 2i$, $w = 2 + i$

c.) $iw = i(2+i) = 2i + i^2 = \boxed{-1+2i}$

e.) $2z - 3w = 2(5-2i) - 3(2+i) = 10 - 4i - 6 - 3i = \boxed{4 - 7i}$

g.) $w^2 = (2+i)^2 = (2+i)(2+i) = 4 + 2i + 2i + i^2 = 4 + 4i - 1 = \boxed{3+4i}$

2.) $z = 1 + i$, $w = -2 + 3i$

c.) $z^3 = (1+i)^3 = (1+i)(1+i)(1+i) = (1 + i + i + i^2)(1+i)$
$\Rightarrow (1 + 2i - 1)(1+i) \Rightarrow 2i(1+i) = 2i + 2i^2 = \boxed{-2 + 2i}$

h.) $izw = i(1+i)(-2+3i) \Rightarrow (i-1)(-2+3i) \Rightarrow -2i + 3i^2 + 2 - 3i$
$\Rightarrow -3 + 2 - 5i \Rightarrow \boxed{-1 - 5i}$

3.) Prove that the sum of a complex number and its conjugate is always real.

If $z = a + bi$, $z^* = a - bi$ $\therefore$ $z + z^* = (a+bi) + (a-bi)$
$\Rightarrow (a+a) + (b-b)i$
$\Rightarrow 2a + 0i$
$z + z^* = 2a$
which is a real number

6.) $z = 2-i$, $w = 1+3i$

b.) $\frac{i}{z} = \frac{i}{(2-i)}\frac{(2+i)}{(2+i)} \Rightarrow \frac{2i+i^2}{4-i^2} \Rightarrow \frac{-1+2i}{4+1} \Rightarrow \frac{-1+2i}{5} \Rightarrow \boxed{\frac{-1}{5}+\frac{2}{5}i}$

d.) $\frac{z^2}{w-i} = \frac{(2-i)^2}{(1+3i)-i} \Rightarrow \frac{(2-i)(2-i)}{1+2i} \Rightarrow \frac{4-2i-2i+i^2}{1+2i} \Rightarrow \frac{(3-4i)}{(1+2i)}\frac{(1-2i)}{(1-2i)}$

$\Rightarrow \frac{3-6i-4i+8i^2}{1-4i^2} \Rightarrow \frac{3-8-10i}{1+4} \Rightarrow \frac{-5-10i}{5} \Rightarrow \boxed{-1-2i}$

7c.) $\frac{1}{2-i} - \frac{2}{2+i}$ Convert to common denominator by multiplying each expression by the opposite denominator

$\Rightarrow \frac{(2+i)}{(2+i)}\frac{1}{(2-i)} - \frac{2}{(2+i)}\frac{(2-i)}{(2-i)} \Rightarrow \frac{2+i}{4-i^2} - \frac{4-2i}{4-i^2}$

$\Rightarrow \frac{2+i}{5} - \frac{4-2i}{5} \Rightarrow \frac{(2+i)-(4-2i)}{5} \Rightarrow \frac{2-4+i+2i}{5} \Rightarrow \frac{-2+3i}{5}$

$\Rightarrow \boxed{\frac{-2}{5}+\frac{3}{5}i}$

11.) $w = \frac{z-1}{z^*+1}$ where $z = a+bi$ for $a, b \in \mathbb{R}$ & z^* is conjugate

a.) write w in the form $x+yi$

$w = \frac{a+bi-1}{a-bi+1} \Rightarrow \frac{(a-1)+bi}{(a+1)-bi} \Rightarrow \frac{((a-1)+bi)}{((a+1)-bi)}\frac{((a+1)+bi)}{((a+1)+bi)}$

by combining "like" terms

$\Rightarrow \frac{(a+1)(a-1)+(a-1)bi+(a+1)bi+b^2i^2}{(a+1)^2-b^2i^2}$

$\Rightarrow \frac{a^2-1+abi-\cancel{bi}+abi+\cancel{bi}-b^2}{(a+1)^2+b^2} \Rightarrow \frac{a^2-b^2-1+2abi}{(a+1)^2+b^2}$

$Re(x)$ $Im(y)$

$\therefore w = \dfrac{(a^2-b^2-1) + 2abi}{(a+1)^2+b^2}$

$$\boxed{\therefore w = \frac{a^2-b^2-1}{(a+1)^2+b^2} + \frac{2ab}{(a+1)^2+b^2}i}$$

b.) w is pure imaginary where $x=0$

$\therefore$ when $\dfrac{a^2-b^2-1}{(a+1)^2+b^2} = 0$

$\therefore$ this would be true when $\boxed{a^2-b^2=1}$

Also, there must be an imaginary number remaining which means $2ab \neq 0$

$\therefore$ $\boxed{a \neq 0 \text{ and } b \neq 0}$